Hyperfocal subalgebras of blocks with Frobenius hyperfocal inertial quotients

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Notation

- Let p be a prime number and $(\mathcal{K}, \mathcal{O}, k)$ a complete p-modular system. We assume that k is an algebraically closed field of characteristic p. In addition, $\mathcal{K}$ is also assumed to be big enough for all finite groups that we consider below.
- Let G be a finite group and b a block of G over $\mathcal{O}$ with a maximal b-Brauer pair (P, b_P). For any $U \leq P$, (U, b_U) denotes the unique b-Brauer pair contained in (P, b_P).
- The group G can act on the set consisting of all b-Brauer pairs by conjugation. For any b-Brauer pair (U, b_U), we denote by $N_G(U, b_U)$ its stabilizer under this action.

Motivation

In order to offer a more general structural explanation for the blockwise Alperin weight conjecture, [1] Rouquier generalized Broué's abelian defect group conjecture by using the **hyperfocal subgroup** of the defect group and proposed the so-called **Rouquier's conjecture**.

[1] R. Rouquier, Block theory via stable and Rickard equivalences, in: Modular Representation Theory of Finite Groups, Charlottesville, VA, 1998, de Gruyter, Berlin, 2001, pp. 101-146.

Hyperfocal subgroup and Rouquier's conjecture

- [1]**Definition of the hyperfocal subgroup**.
  Denote by D the subgroup of P generated by the subsets $\{uxu^{-1}x^{-1} \mid u \in U,\ x \in N_G(U, b_U)_{p'}\}$, where U runs on the set of subgroups of P. The subgroup D is called the *hyperfocal subgroup* of P.

- **Rouquier's conjecture**.
  When D is abelian, the two block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(D)c$ are basically Rickard equivalent.

  Here, the block c is the Brauer correspondent of the block b in $N_G(D)$.

[1] L. Puig, The hyperfocal subalgebra of a block, *Invent. Math.* 141, 365-397, 2000.

In the light of Rouquier's conjecture, we can investigate the structure of block algebras in terms of hyperfocal subgroups.

[1] **Theorem 1**. Broué's abelian defect group conjecture holds for blocks with nontrivial cyclic hyperfocal subgroups.

- The key step of the proof is that we showed that the so-called hyperfocal subalgebra of the block with abelian defect group P and nontrivial cyclic hyperfocal subgroup D is a **Brauer tree algebra**.

[1] X. Hu, K. Zhang and Y. Zhou, Broué's abelian defect group conjecture for blocks with cyclic hyperfocal subgroups, *Bull. London Math. Soc.* 56, 2188-2211, 2024.

Hyperfocal subalgebra

Recall the definition of the source algebra.

- The defect group P acts on the $\mathcal{O}$-algebra $\mathcal{O}Gb$ via the conjugation. This action gives a subalgebra $(\mathcal{O}Gb)^P = \{a \in \mathcal{O}Gb \mid vav^{-1} = a, \forall v \in P\}$.
- Let i be a primitive idempotent of $(\mathcal{O}Gb)^P$ such that $\mathrm{Br}_P(i) \neq 0$. Here, Br_P is the Brauer homomorphism from $(\mathcal{O}Gb)^P \to kC_G(P)$.
- The $\mathcal{O}$-algebra $i\mathcal{O}Gi$, denoted by A, is a **source algebra** of the block b.

Hyperfocal subalgebra (cont.)

- [1]**Definition of the hyperfocal subalgebra**.
  There is a unitary P-stable $\mathcal{O}$-subalgebra $\mathbb{A}$ of A, unique up to $(A^P)^\times$-conjugation, containing D and fulfilling $A = \oplus_{u\in P/D}\mathbb{A}\cdot u$. The subalgebra $\mathbb{A}$ is called a *hyperfocal subalgebra* of the block b.

- **Problem**: What's the structure of the hyperfocal subalgebra $\mathbb{A}$ when the hyperfocal subgroup D is nontrivial and cyclic in general?

[1] L. Puig, The hyperfocal subalgebra of a block, *Invent. Math.* 141, 365-397, 2000.

- In fact, we can deal with this problem in a unified framework which includes cyclic hyperfocal subgroups and Klein four hyperfocal subgroups.
- In order to introduce this framework, we need the following result that leads to the definition of **hyperfocal inertial quotients**.

Hyperfocal inertial quotients

Proposition 2. Let D be a hyperfocal subgroup. Then $N_G(D, b_D)/DC_G(D)$ is a p-nilpotent group, i.e., it has a normal p-complement.

- We call the normal p-complement of $N_G(D, b_D)/DC_G(D)$ the **hyperfocal inertial quotient** of the block b and denote it by $E_{\mathfrak{h}}$.

Frobenius hyperfocal inertial quotient

The unified framework is as follows.

- The block b with **Frobenius hyperfocal inertial quotient** means that D is nontrivial abelian and the semidirect product $D \rtimes E_{\mathfrak{h}}$ is a Frobenius group. The hyperfocal subalgebra of this block is called a **Frobenius hyperfocal subalgebra** for simplicity.
- It is an analogue of the block with abelian defect groups and Frobenius inertial quotients.

Main Theorem

Main Theorem. Suppose that the block b is with Frobenius hyperfocal inertial quotient. Then there is a stable equivalence of Morita type between the hyperfocal subalgebra $\mathbb{A}$ of the block b and the group algebra $\mathcal{O}(D \rtimes E_{\mathfrak{h}})$.

Applications: algebra structures

Application to cyclic hyperfocal subgroups.

Proposition 3. Suppose that the hyperfocal subgroup D is a nontrivial cyclic group. Then the hyperfocal subalgebra $\mathbb{A}$ is Rickard equivalent to $\mathcal{O}(D \rtimes E_{\mathfrak{h}})$.

Applications: algebra structures (cont.)

Application to hyperfocal subgroups being Klein four groups.

Proposition 4. Suppose that the hyperfocal subgroup D is a Klein four group. Then the hyperfocal subalgebra $\mathbb{A}$ is Morita equivalent either to $\mathcal{O}(D \rtimes E_{\mathfrak{h}})$ or the principal block of A_5. In particular, $\mathbb{A}$ is Rickard equivalent to $\mathcal{O}(D \rtimes E_{\mathfrak{h}})$.

As a corollary, we can show that Broué's abelian defect group conjecture holds for **blocks with Klein four hyperfocal subgroups**.

Applications: character structures

First, we recall a conjecture proposed by Kessar, Linckelmann and Navarro, which is viewed as a 'hyperfocal version' of Brauer's height zero conjecture and referred to the KLN conjecture here.

[1] **The KLN conjecture**.
Suppose that $\mathcal{K}$ is a splitting field of the semisimple $\mathcal{K}$-algebra $\hat{\mathbb{A}} = \mathcal{K} \otimes_{\mathcal{O}} \mathbb{A}$. Then all simple $\hat{\mathbb{A}}$-modules have dimensions coprime to p if and only if the hyperfocal subgroup D is abelian.

[1] R. Kessar, M. Linckelmann and G. Navarro, A characterization of nilpotent blocks, *Proc. Amer. Math. Soc.* 143, 5129-5138, 2015.

Applications: character structures (cont.)

Proposition 5. The forward direction of the KLN conjecture holds when the hyperfocal subgroup is either a cyclic group or a Klein four group.

Thank you!